Exercises from Atiyah-MacDonald
Introduction to Commutative Algebra

Joshua Ruiter

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Exercises: 2.12, 5.1, 5.4, 5.8, 5.9, 5.10, 5.12, 5.13, 5.14, 5.16 of Atiyah-MacDonald, and one extra exercise from Professor Pappas

Proposition 0.1 (Exercise 2.12). *Let M be a finitely generated A-module and $\phi : M \to A^n$ a surjective homomorphism. Then $\ker \phi$ is finitely generated.*

Proof. Since A^n is a free, and hence projective, A-module, the short exact sequence

$$0 \to \ker \phi \to M \to A^n \to 0$$

splits, so $M \cong A^n \oplus \ker \phi$. Since M and A^n are finitely generated, so is $\ker \phi$. □

Proof. (Alternate proof of 0.1) Let $e_1, \ldots, e_n$ be a basis of A^n, and let $u_1, \ldots, u_n \in M$ so that $\phi(u_i) = e_i$. Let $N \subset M$ be the submodule generated by $u_1, \ldots, u_n$. Since $M/\ker \phi \cong A^n$, every element of M can be written as linear combination of the u_i plus some element of ϕ, thus $M = N + \ker \phi$. This sum is direct since if $x = \sum a_i u_i \in N \cap \ker \phi$, then

$$\phi(x) = \sum_i a_i e_i = 0 \qquad \implies \qquad a_i = 0 \ \forall i \qquad \implies \qquad x = 0$$

since $e_1, \ldots, e_n$ is a basis of A_n. Thus $M = N \oplus \ker \phi$. Since M, N are finitely generated, $\ker \phi$ is finitely generated. □

Recall for the next proposition that the sets

$$V(I) = \{\mathfrak{p} \in \operatorname{spec} A : I \subset \mathfrak{p}\}$$

are the closed sets of the Zariski topology on $\operatorname{spec} A$, where $I \subset A$ is any ideal.

Lemma 0.2 (for Exercise 5.1). *Let $f : A \to B$ be a ring homomorphism, with induced map $f^* : \operatorname{spec} B \to \operatorname{spec} A$. For any ideal $I \subset B$,*

$$f^*(V(I)) \subset V(f^{-1}(I))$$

with equality if f is integral.

Proof. The first inclusion is immediate: if $\mathfrak{p} \in f^*(V(I))$, then $I \subset \mathfrak{p}$, so $f^{-1}(I) \subset f^{-1}(\mathfrak{p})$.

Now we show the reverse inclusion, assuming f is integral. Let $\mathfrak{p} \in V(f^{-1}(I))$, so $f^{-1}(I) \subset \mathfrak{p}$. We need to show that $\mathfrak{p} \in f^*(V(I))$, so we need to find $\mathfrak{q} \in \operatorname{spec} B$ so that $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q}) = \mathfrak{p}$. Consider the factorization

$$A \xrightarrow{f} f(A) \hookrightarrow B$$

If $a \in \ker f$, then $f(a) = 0 \in I$, so $a \in f^{-1}(I) \subset \mathfrak{p}$. Thus $\ker f \subset \mathfrak{p}$, so by the ideal correspondence (Proposition 1.1 of Atiyah-MacDonald) and a remark on page 9 of Atiyah-MacDonald, $\mathfrak{p}$ corresponds to a prime ideal $\mathfrak{p}'$ of $f(A)$, with $f^{-1}(\mathfrak{p}') = \mathfrak{p}$. Since B is integral over $f(A)$, by the going-up theorem (Theorem 5.10 of Atiyah-MacDonald), there exists $\mathfrak{q} \in \operatorname{spec} B$ so that $\mathfrak{q} \cap f(A) = \mathfrak{p}'$. Then

$$f^{-1}(\mathfrak{q}) = f^{-1}\Big(\mathfrak{q} \cap f(A)\Big) = f^{-1}(\mathfrak{p}') = \mathfrak{p}$$

We include the following commutative diagram as a visual aid.

$$\begin{array}{ccccc} A & \xrightarrow{f} & f(A) & \hookrightarrow & B \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{p} & \longrightarrow & \mathfrak{p}' & \hookrightarrow & \mathfrak{q} \end{array}$$

□

Proposition 0.3 (Exercise 5.1)**.** *Let $f : A \to B$ be an integral homomorphism of rings. Then $f^* : \operatorname{spec} B \to \operatorname{spec} A$ is a closed mapping.*

Proof. Any closed subset of $\operatorname{spec} B$ is of the form $V(I)$, and by Lemma 0.2, the image of $V(I)$ under f^* is $V(f^{-1}(I))$, which is closed. □

Exercise 5.4. Let $A \subset B$ be a subring, with B integral over A. Let $\mathfrak{n} \subset B$ be a maximal ideal, and $\mathfrak{m} = A \cap \mathfrak{n}$ the corresponding maximal ideal of A. In general, it is not the case that $B_\mathfrak{n}$ is integral over $A_\mathfrak{m}$. We provide an explicit counterexmple, following the hint in Atiyah-MacDonald.

Let k be a field, $B = k[x], A = k[x^2 - 1]$. Let $\mathfrak{n}$ be the maximal ideal generated by $(x-1)$, and $\mathfrak{m} = A \cap \mathfrak{n}$. Note that $\mathfrak{m} = (x^2 - 1)$, which is the unique maximal ideal of A, so $A_\mathfrak{m} = A$. We can write $B_\mathfrak{n}$ as

$$B_\mathfrak{n} = \left\{ \frac{f}{g} \in k(x) \;\middle|\; (x-1) \nmid g \right\}$$

In particular, $B_\mathfrak{n} \neq B$. We have inclusions

$$A_\mathfrak{m} = A \hookrightarrow B = k[x] \hookrightarrow B_\mathfrak{n}$$

Since $k[x]$ is integrally closed (see comment on pages 62-63 of Atiyah-MacDonald), $B_\mathfrak{n}$ cannot be integral over $k[x]$. Thus $B_\mathfrak{n}$ cannot be integral over $A_\mathfrak{m}$.

Proposition 0.4 (Exercise 5.8i). *Let A be a subring of an integral domain B and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ so that $fg \in C[x]$. Then $f, g \in C[x]$.*

Proof. Let $f, g \in B[x]$ be monic with $fg \in C[x]$. Let K be the fraction field of B, and let $\overline{K}$ be an algebraic closure of K. Note that by Corollary 5.5 of Atiyah-MacDonald, C is integrally closed in $\overline{K}$. Over $\overline{K}$, f, g split into linear factors,

$$f(x) = \prod_i (x - \xi_i) \qquad g(x) = \prod_j (x - \eta_j)$$

Then ξ_i, η_j are roots of $fg \in C[x]$, which is monic, so by Proposition 5.15, $\xi_i, \eta_j \in \overline{K}$ are integral over C. Since the integral elements form a ring and all other coefficients of f (g) are polynomials in the ξ_i (η_j), the coefficients of f (g) lie in the integral closure of C in $\overline{K}$, which is C since C is integrally closed. That is, $f, g \in C[x]$. □

Remark 0.5. The next proposition is the same as the previous, except that the hypothesis of being an integral domain is removed.

Proposition 0.6 (Exercise 5.8ii). *Let $A \subset B$ be a subring and let C be the integral closure of A in B. Let f, g be monic polynomials in $B[x]$ so that $fg \in C[x]$. Then $f, g \in C[x]$.*

Proof. Not assigned. □

Lemma 0.7 (for Exercise 5.9). *Let $A \subset C$ be rings. If C is integral over A, then $C[x]$ is integral over $A[x]$.*

Proof. Let $f \in C[x]$. To show that f is integral over $A[x]$, using the criterion of Proposition 5.1 of Atiyah-MacDonal, it suffices to show that $A[x, f]$ is contained in a finitely generated $A[x]$-module. Let $c_1, \ldots, c_n$ be the coefficients appearing in f. Then $A[x, f] \subset A[x, c_1, \ldots, c_n] = A[x][c_1, \ldots, c_n]$. Since c_i are all integral over A, by the same criterion, $A[c_1, \ldots, c_n]$ is a finitely generated A-module. Thus $A[x][c_1, \ldots, c_n]$ is a finitely generated $A[x]$-module. □

Proposition 0.8 (Exercise 5.9). *Let $A \subset B$ be a subring, and let C be the integral closure of A in B. Then $C[x]$ is the integral closure of $A[x]$ in $B[x]$.*

Proof. By Lemma 0.7, $C[x]$ is integral over $A[x]$, so it suffices to show that any element of $B[x]$ integral over $A[x]$ lies in $C[x]$. Let $f \in B[x]$ be integral over $A[x]$, so we have an equation

$$f^m + g_1 f^{m-1} + \ldots + g_m = 0$$

with $g_i \in A[x]$. Let $r = \max\{m, \deg g_1, \ldots, \deg g_n\}$, and set $f_1 = f - x^r$. Substituting $f_1 - x^r$ into our previous equation, we obtain

$$(f_1 - x^r)^m + g_1(f_1 - x^r)^{n-1} + \ldots + g_m = 0$$

which we then expand as a polynomial in f_1, to obtain

$$f_1^m + h_1 f_1^{m-1} + \ldots + \ldots h_m = 0 \tag{0.1}$$

with $h_i \in A[x]$. In particular,

$$h_m = (x^r)^m + g_1(x^r)^{m-1} + \ldots + g_m \in A[x]$$

Rearranging equation 0.1, we obtain

$$h_m = -f_1^m - h_1 f_1^{m-1} + \ldots - h_{m-1} f_1 = -f_1 \Big(f_1^{m-1} + h_1 f_1^{m-2} + \ldots + h_{m-1} \Big)$$

Note that $-f_1$ and $f_1^{m-1} + h_1 f_1^{m-2} + \ldots + h_{m-1}$ are monic polynomials in $B[x]$ with product in $A[x] \subset C[x]$. Hence proposition 0.6 applies, so we conclude that $f_1 \in C[x]$. In particular, $f = f_1 - x^r \in C[x]$. Thus $C[x]$ is the integral closure of $A[x]$ in $B[x]$. □

Proposition 0.9 (Exercise 5.10i)**.** *Let $f : A \to B$ be a ring homomorphism, and $f^* : \operatorname{spec} B \to \operatorname{spec} A$ the induced map. Among the following statements, (1) $\implies$ (2) $\iff$ (3).*

1. *f^* is a closed map.*
2. *f has the going-up property.*
3. *Let $\mathfrak{q} \in \operatorname{spec} B$, and $\mathfrak{p} = f^*(\mathfrak{q})$. Then $f^* : \operatorname{spec}(B/\mathfrak{q}) \to \operatorname{spec}(A/\mathfrak{p})$ is surjective.*

Proof. I don't know how to prove this. □

Proposition 0.10 (Exercise 5.10ii)**.** *Let $f : A \to B$ be a ring homomorphism, and $f^* : \operatorname{spec} B \to \operatorname{spec} A$ the induced map. Among the following statements, (1) $\implies$ (2) $\iff$ (3).*

1. *f^* is an open map.*
2. *f has the going-down property.*
3. *Let $\mathfrak{q} \in \operatorname{spec} B$, and $\mathfrak{p} = f^*(\mathfrak{q})$. Then $f^* : \operatorname{spec}(B_{\mathfrak{q}}) \to \operatorname{spec}(A_{\mathfrak{q}})$ is surjective.*

Proof. I don't know how to prove this. □

Proposition 0.11 (Exercise 5.12i)**.** *Let G be a finite group of automorphisms of a ring A, and let A^G be the subring of G-invariants. Then A is integral over A^G.*

Proof. For $a \in A$, the monic polynomial

$$f(x) = \prod_{\sigma \in G} (x - \sigma a)$$

has a as a root. Note that the G-action on A induces an action on $A[x]$ by fixing x and acting on the coefficients, so $A[x]^G = A^G[x]$. We claim that $f \in A^G[x]$, since for $\tau \in G$,

$$\tau f(x) = \tau \prod_{\sigma \in G} (x - \sigma a) = \prod_{\sigma \in G} (x - \tau\sigma a) = \prod_{\sigma \in G} (x - \sigma a) = f$$

Thua a is integral over A^G. □

Proposition 0.12 (Exercise 5.12ii)**.** *Let G be a finite group of automorphisms of a ring A, and let A^G be the subring of G-invariants. Let $S \subset A$ be a multiplicative set which is stable under G, that is, $\sigma(S) \subset S$ for $\sigma \in G$. The G-action on A extends to a G-action on $S^{-1}A$ via*

$$\sigma\left(\frac{a}{s}\right) = \frac{\sigma a}{\sigma s}$$

Let $S^G = S \cap A^G$. Then we have an isomorphism

$$(S^G)^{-1}A^G \to (S^{-1}A)^G \qquad \frac{a}{s} \mapsto \frac{a}{s}$$

Proof. We check that the G-action on $S^{-1}A$ is well defined. Suppose $\frac{a}{s} = \frac{a'}{s'}$. We need to verify that $\sigma\left(\frac{a}{s}\right) = \sigma\left(\frac{a'}{s'}\right)$. By the previous equality, $\exists t \in S$ so that $t(as' - a's) = 0$. Then since σ is an automorphism and S is stable under σ

$$(\sigma t)\Big((\sigma a)(\sigma s') - (\sigma a')(\sigma s)\Big) = 0$$

thus $\sigma\left(\frac{a}{s}\right) = \sigma\left(\frac{a'}{s'}\right)$, so σ is well-defined. It is then clear that σ is an automorphism of $S^{-1}A$, and this extends the G-action in the sense that $\sigma\left(\frac{a}{1}\right) = \frac{\sigma a}{1}$. (Recall that $A \to S^{-1}A$ may not be injective if S contains zero divisors, so we cannot in general regard A as a subring of $S^{-1}A$.)

Before we verify the isomorphism $(S^G)^{-1}A \cong (S^{-1}A)^G$, note that S^G is a multiplicative set, since for $s, t \in S^G$, we have $s, t \in S \implies st \in S$, and

$$s, t \in A^G \implies \sigma s = s, \sigma t = t \implies \sigma(st) = (\sigma s)(\sigma t) = st \implies st \in A^G$$

If $\frac{a}{s} \in (S^G)^{-1}A^G$, where $a \in A^G, s \in S^G$, then by definition of the group action on $S^{-1}A$ we have $\sigma\left(\frac{a}{s}\right) = \frac{\sigma a}{\sigma s} = \frac{a}{s}$, so $\frac{a}{s} \in (S^{-1}A)^G$. It is clear that the map is well-defined and is a homomorphism.

First, we show it is injective. Let $\frac{a}{s} \in (S^G)^{-1}A^G$ be in the kernel, so $\frac{a}{s} = \frac{0}{1}$ in $(S^{-1}A)^G$. Then there exists $t \in S$ so that $ta = 0$. Taking $t' = \prod_{\sigma \in G} \sigma t$, so that $t' \in S^G$, we also get $t'a = 0$, which says that $\frac{a}{s} = \frac{0}{1}$ in $(S^G)^{-1}A^G$.

Now we show that it is surjective. Let $\frac{a}{s} \in (S^{-1}A)^G$, so $\frac{\sigma a}{\sigma s} = \frac{a}{s}$. We meed to fine $a' \in A^G, s' \in S^G$ so that $\frac{a}{s} = \frac{a'}{s'}$. First, note that

$$\frac{a}{s} = \frac{\left(\prod_{\sigma \neq 1} \sigma s\right) a}{\prod_{\sigma \in G} \sigma s}$$

where the denominator is in S^G, so we may assume $s \in S^G$, that is, $\frac{\sigma a}{s} = \frac{a}{s}$. So for $\sigma \in G$, there is t_σ so that

$$t_\sigma(\sigma a - a) = 0$$

Set $t = \prod_{\sigma \in G} t_\sigma$, so $t(\sigma a - a) = 0$ for all $\sigma \in G$. Set $v = \prod_{\sigma \in G} \sigma t$, so $v(\sigma a - a) = 0$ for all $\sigma \in G$, and $v \in S^G$. Then

$$va = v(\sigma a) = (\sigma v)(\sigma a) = \sigma(va) \implies va \in A^G$$

Thus $\frac{a}{s} = \frac{va}{vs}$, where $va \in A^G$ and $vs \in S^G$. Thus our map is surjective. This completes the proof that our map is an isomorphism. □

Proposition 0.13 (Exercise 5.13). *Let A be a ring, and G a finite group of automophisms of A. Let $\mathfrak{p} \subset A^G$ be a prime ideal, and let P be the set of prime ideals of A whose contraction is $\mathfrak{p}$. Then G acts transitively on P. In particular, P is finite.*

Proof. First, it is clear that G acts on P, since G is a group of automorphisms of A. Let $\mathfrak{q}_1, \mathfrak{q}_2 \in P$, and $x \in \mathfrak{q}_1$. Then

$$\prod_{\sigma \in G} \sigma x \in \mathfrak{q}_1 \cap A^G \subset \mathfrak{q}_2$$

Then since $\mathfrak{q}_2$ is prime, $\sigma x \in \mathfrak{q}_2$, or equivalently $x \in \sigma^{-1}\mathfrak{q}_2$, for some $\sigma \in G$. Thus

$$\mathfrak{q}_1 \subset \bigcup_{\sigma \in G} \sigma^{-1}\mathfrak{q}_2 = \bigcup_{\sigma \in G} \sigma\mathfrak{q}_2$$

Then by Proposition 1.11 of Atiyah-MacDonald, $\mathfrak{q}_1 \subset \sigma\mathfrak{q}_2$ for some $\sigma \in G$. Then since $\mathfrak{q}_1, \sigma\mathfrak{q}_2$ are both ideals whose contraction is $\mathfrak{p}$, by Corollary 5.9 of Atiyah-MacDonald, $\mathfrak{q}_1 = \sigma\mathfrak{q}_2$. Thus G acts transively on P. Thus for any $\mathfrak{q} \in P$,

$$P = \{\sigma\mathfrak{q} : \sigma \in G\}$$

which is finite since G is finite. □

Proposition 0.14 (Exercise 5.14). *Let A be an integrally closed domain with fraction field K, and let L be a finite Galois extension of K. Let $G = \mathrm{Gal}(L/K)$, and let B be the integral closure of A in L. Then $\sigma B = B$ for all $\sigma \in G$, and $A = B^G$.*

Proof. Let $\beta \in B$. Since β is integral over A, it satisfies a monic polynomial with coefficients in A,

$$\beta^n + a_{n-1}\beta^{n-1} + \ldots + a_0 = 0 \qquad a_i \in A$$

Applying $\sigma \in G$ to this equation, the a_i are fixed, since $A \subset K$, so we obtain

$$(\sigma\beta)^n + a_{n-1}(\sigma\beta)^{n-1} + \ldots + a_0 = 0$$

which is again a monic polynomial with coefficients in A, so $\sigma\beta$ is integral over A, and $\sigma B \subset B$. Thus $\sigma\beta \in B$. Performing an identical argument by applying σ^{-1} instead show that $B \subset \sigma B$, hence $B = \sigma B$.

We know $A \subset B$ and $A \subset K = L^G$, so clearly $A \subset B^G$. For the other inclusion, any $\beta \in B^G$ is in $L^G = K = \mathrm{Frac}(A)$, and since $\beta \in B$, β is integral over A. Since A is integrally closed in K, $\beta \in A$, hence $B^G \subset A$. Thus $A = B^G$. □

Proposition 0.15 (roughly a converse to Exercise 5.14)**.** *Let A be an integral domain with fraction field K, and let L be a finite Galois extension of K. Let $G = \mathrm{Gal}(L/K)$, and let B be the integral closure of A in L. If $A = B^G$, then A is integrally closed.*

Proof. We need to show that if $\alpha \in K$ is integral over A, then $\alpha \in A$. Let $\alpha \in K$ be integral over A. Since $\alpha \in L$ and α is integral over A, $\alpha \in B$. By Galois theory, $K = L^G$ so $\alpha \in B \cap L^G = B^G$. By hypothesis, $A = B^G$, so $\alpha \in A$. □

Proposition 0.16 (Extra exercise, part 1)**.** *Let $n \in \mathbb{Z}_{\geq 1}$, $A = \mathbb{C}[x, y]$, and let ζ be a primitive nth root of unity. We define a $\mathbb{C}$-linear action of $G = \mathbb{Z}/n\mathbb{Z}\langle\sigma\rangle$ on A by*

$$\sigma x = \zeta x \qquad \sigma y = \zeta^{-1} y$$

Let $B = \mathbb{C}[u, v, w]/(uv - w^n)$. The $\mathbb{C}$-algebra homomorphism defined by

$$\phi : \mathbb{C}[u, v, w] \to A^G \qquad u \mapsto x^n,\ v \mapsto y^n,\ w \mapsto xy$$

is surjective and has kernel $(uv - w^n)$, so it induces an isomorphism of $\mathbb{C}$-algebras $B \cong A^G$.

Proof. First, note that ϕ is well defined, since $x^n, y^n, xy \in A^G$. First, we show that ϕ is surjective. Let $f \in A^G \subset \mathbb{C}[x, y]$,

$$f(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + \ldots = a_0 + a_1 \zeta x + a_2 \zeta^{-1} y + a_3 \zeta^2 x^2 + \ldots$$

Matching up the like terms, we obtain equations

$$a_1 = \zeta a_1 \qquad a_2 = \zeta^{-1} a_2 \qquad a_3 = \zeta^2 a_3 \qquad \ldots$$

which imply that any coefficients except for $1, x^n, y^n, xy$ are zero. Thus $A^G = \mathbb{C}[x^n, y^n, xy]$, so ϕ is surjective. Now we consider the kernel. Clearly, $uv - w^n \in \ker \phi$, since

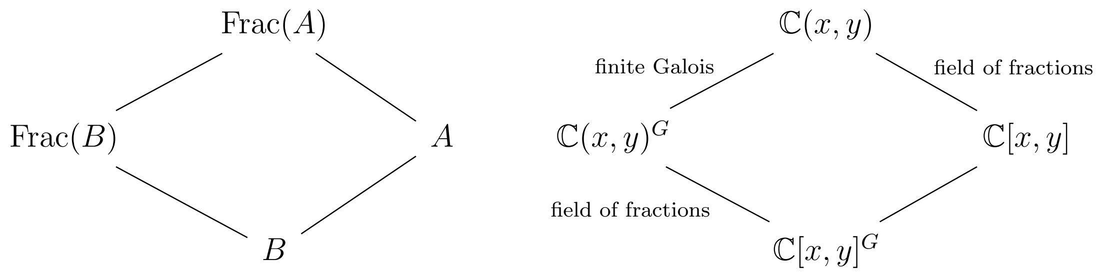
$$\phi(uv - w^n) = \phi(u)\phi(v) - \phi(w)^n = x^n y^n - (xy)^n = 0$$

The kernel must be an ideal of $\mathbb{C}[u, v, w]$, and $\mathbb{C}[u, v, w]$ is PID, so $\ker \phi$ is a principal ideal, so it must be generated by somthing dividing $uv - w^n$. But $uv - w^n$ is irreducible, so $\ker \phi = (uv - w^n)$. Thus by the first isomoprhism theorem, we get an induced isomorphism $B \cong A^G$. □

Proposition 0.17 (Extra exercise, part 2)**.** *Let $B = \mathbb{C}[u, v, w]/(uv - w^n)$. Then B is integrally closed (in its own fraction field).*

Proof. It is clear that B is an integral domain, since $uv - w^n$ is irreducible, or alternately, via the isomorphism above, B is (isomorphic to) a subring of the integral domain $\mathbb{C}[x, y]$.

Let $A = \mathbb{C}[x, y]$ and G be as in the previous proposition, and we use the isomorphism to identify B with A^G. Note that $\mathrm{Frac}(A) = \mathbb{C}(x, y)$. By Proposition 0.12, $\mathrm{Frac}(B) = \mathrm{Frac}(A)^G = \mathbb{C}(x, y)^G$. By Proposition 0.11, A is integral over $B = A^G$. We know that any polynomial ring over a field is integrally closed, so A is the integral closure of B in L. Then all the hypotheses of Proposition 0.15 are satisfied, so B is integrally closed. We depict the situation below in two alternate notations.



□